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Higher Order Ito Product Formula and Generators of Evolutions and Flows

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A simple combinatorial formula is found for the product of two iterated quantum stochastic integrals, and used to find conditions that such an integral represent a unitary-valued or *-algebra homomorphism-valued process.

1. HIGHER ORDER ITO PRODUCT FORMULA

The integrators of multidimensional quantum stochastic calculus can be parametrized by elements of the space $\mathcal{J}$ consisting of linear transformations H in the finite-dimensional Hilbert space $\mathbb{C} \oplus \mathcal{K}$. Such a transformation decomposes naturally (Parthasarathy, 1992) into four components comprising a complex number, a vector in $\mathcal{K}$, a linear form on $\mathcal{K}$, and a linear transformation on $\mathcal{K}$, corresponding to the time, creation, annihilation, and multidimensional gauge components of the integrator. The corresponding integrator process Λ^H consists of operators in the Fock space $\mathcal{H} = \Gamma(L^2(\mathbb{R}_+) \otimes \mathcal{K})$ whose matrix elements between exponential vectors are given by

$$\langle e(f), \Lambda_t^H e(g)\rangle = \int_0^t \langle \tilde{f}(s), H\tilde{g}(s)\rangle\, ds\, \langle e(f), e(g)\rangle$$

where for $u \in \mathcal{K}$, $\tilde{u} = (1, u) \in \mathbb{C} \oplus \mathcal{K}$. We have $(\Lambda_t^H)^\dagger = \Lambda_t^{H^\dagger}$ where $H^\dagger$ is the usual adjoint and $(\Lambda_t^H)^\dagger$ the restriction of the adjoint to the exponential domain. By the quantum Ito formula (Hudson and Parthasarathy, 1984) we have

$$d\Lambda_t^H\, d\Lambda_t^K = d\Lambda_t^{H \circ K} \tag{1.1}$$

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where

$$H \circ K = HEK$$

E being the projector on the canonical embedding of $\mathcal{H}$ in $\mathbb{C} \oplus \mathcal{H}$. When equipped with the usual adjunction and the associative multiplication $\circ$ we call $\mathcal{I}$ the *Ito algebra*. By embedding them in the Fock space (Parthasarathy, 1992), product formulas for other stochastic calculus theories correspond to subalgebras of $\mathcal{I}$.

In this paper we are interested in iterated integrals such as

$$I_t(H_1, \ldots, H_m) = \int_{0<t_1<\cdots<t_m<t} d\Lambda^{H_1}(t_1) \cdots d\Lambda^{H_m}(t_m)$$

Note that

$$I_t(H_1^\dagger, \ldots, H_m^\dagger) = I_t(H_1, \ldots, H_m)^\dagger \tag{1.2}$$

There is a product formula for such integrals expressed in the following theorem, which is proved using (1.1) by induction on $m + n$ [the case of purely gauge integrals was given in Hudson and Parthasarathy (1993)].

Theorem. We have

$$\begin{aligned} &I(H_1, \ldots, H_m)I(H_{m+1}, \ldots, H_{m+n}) \\ &= \sum_{r=\max\{m,n\}}^{m+n} \sum_{P \in \mathbb{P}_r} I(H_{P_1}, \ldots, H_{P_r}) \end{aligned} \tag{1.3}$$

where $\mathbb{P}_r$ is the set of ordered partitions $P = (P_1, \ldots, P_r)$ of $(1, \ldots, m + n)$ into r subsets which are either singletons or pairs $\{p, q\}$ with $p \in \{1, \ldots, m\}$ and $q \in \{m + 1, \ldots, m + n\}$ in which $\{1, \ldots, m\}$ and $\{m + 1, \ldots, m + n\}$ occur in their natural orders, and $H_{\{p,q\}} = H_p \circ H_q$.

Since $I(H_1, \ldots, H_m) = I(H_1 \otimes \cdots \otimes H_m)$ is linear in $H_1, \ldots, H_m$, we may extend I to a linear map from elements of the tensor space

$$\mathcal{T} = \mathbb{C} \oplus \mathcal{I} \oplus \mathcal{I} \otimes \mathcal{I} \oplus \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I} \oplus \cdots$$

over $\mathcal{I}$ to processes in Fock space, in such a way that

$$I(A)I(B) = I(A * B), \qquad I(A^\dagger) = I(A)^\dagger \tag{1.4}$$

where the associative multiplication $*$ is determined by (1.3) and the involution on tensors is inherited from that on $\mathcal{I}$. Note also that $*$ is well defined on the extended tensor space (in which infinitely many homogeneous components may be nonzero) even though the integral I may no longer be defined. We denote the extended tensor space equipped with the product $*$ and the involution $\dagger$ inherited from $\mathcal{I}$ by $\Gamma(\mathcal{I})$.

In most applications of quantum stochastic calculus there is given an *initial* unital †-algebra $\mathscr{A}$. We define the multiplication $*$ in the tensor product $\tilde{\Gamma}(\mathscr{I}) = \mathscr{A} \otimes \Gamma(\mathscr{I})$ by the natural product rule $(a \otimes A) * (b \otimes B) = ab \otimes A * B$, and equip it with the product involution. The integration map I ampliates to the tensor product with $\mathscr{A}$, so that (1.4) remains true for integrands in $\mathscr{A} \otimes \Gamma(\mathscr{I})$ when the integrals are defined.

2. EVOLUTION GENERATORS

Let $R = \{k_1 < \cdots < k_r\}$ and $S = \{l_1 < \cdots < l_s\}$ be possibly empty sets whose union is $\{1, \ldots, n\}$, with respectively $r = |R|$ and $s = |S|$ elements. For a nonnegative integer n let $\mathscr{A} \otimes \mathscr{I}^{\otimes n} = \mathscr{A}$ if $n = 0$ and $\mathscr{A} \otimes \mathscr{I} \otimes \cdots \otimes \mathscr{I}$ (n factors) if $n > 0$. Let $a \in \mathscr{A} \otimes \mathscr{I}^{\otimes r}$ and $b \in \mathscr{A} \otimes \mathscr{I}^{\otimes s}$, respectively, and define an element $a^R b^S$ of $\mathscr{A} \otimes \mathscr{I}^{\otimes n}$ by bilinear extension of the rule for product elements:

$$(a^0 \otimes a^1 \otimes \cdots \otimes a^r)^R (b^0 \otimes b^1 \otimes \cdots \otimes b^s)^S = c^0 \otimes c^1 \otimes \cdots \otimes c^n$$

where $c^0 = a^0 b^0$, and for $j = 1, \ldots, n$, c^j is a^i if $j_i = k_i \in R \cap S'$, b^m if $j = l_m \in R' \cap S$, and $a^i \circ b^m$ if $j = k_i = l_m \in R \cap S$ (complements are in $\{1, \ldots, n\}$). Note that by taking either R or S empty we obtain a bimodule action of $\mathscr{A}$ on each $\mathscr{A} \otimes \mathscr{I}^{\otimes n}$. Using (1.3), the product $a * b = c = (c_0, c_1, \ldots)$ of elements of $\tilde{\Gamma}(\mathscr{I})$ can then be expressed in component form as

$$c_n = \sum a_r^R b_s^S \tag{2.1}$$

where, for $n = 0, 1, 2, \ldots$, the sum is over the 3^n choices of ordered pairs of subsets R and S whose union is $\{1, \ldots, n\}$.

Whether or not $I(u)$ exists, conditions on $u \in \tilde{\Gamma}(\mathscr{I})$ formally equivalent to the unitarity of the process $I(u)$ are that

$$u * u^\dagger = u^\dagger * u = (\mathbf{1}, 0, 0, \ldots)$$

where $\mathbf{1}$ is the identity element of $\mathscr{A}$. Evidently such elements form a group G under $*$. Moreover, $u \in G$ if and only if its components satisfy

$$\sum_{R \cup S = \{1,\ldots,n\}} u_r^R u_s^{\dagger S} = \sum_{R \cup S = \{1,\ldots,n\}} u_r^{\dagger R} u_s^S = \delta_{n,0}\mathbf{1}, \qquad n = 0, 1, \ldots \tag{2.2}$$

For $N = 0, 1, \ldots$ let us denote by G_N the set of sequences $(u_0, \ldots, u_N)$, with each $u_n \in \mathscr{A} \otimes \mathscr{I}^{\otimes n}$, and such that (2.2) holds for $n = 0, \ldots, N$. Then G_N is a group under the composition defined by (2.1); we call its elements *Nth-order evolution generators*. Can such a generator $(u_0, \ldots, u_N)$ be extended to an element of G?

This question can be answered affirmatively in the case of purely gauge stochastic integrals (Hudson and Parthasarathy, 1993). Indeed, let $\mathscr{I}_0$ be the subalgebra of $\mathcal{I}$ consisting of linear transformations on $\mathcal{H}$. Then in this case $u_j \in \mathcal{A} \otimes \mathcal{I}_0^{\otimes j}$, $j = 1, \ldots, N$. The projector E of Section 1 is just the identity of $\mathcal{I}_0$, so that $H \circ K = HK$ for $H, K \in \mathcal{I}_0$. We may write the condition on the additional element u_{N+1} that $(u_0, \ldots, u_N, u_{N+1}) \in G_{N+1}$ in the form

$$[\sum u_r^R(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + u_{N+1}][\sum u_r^{\dagger R}(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + u_{N+1}^\dagger] = \mathbf{1} \otimes E^{\otimes N+l} \tag{2.3}$$

together with the corresponding relations with all u's and $u^\dagger$ exchanged. In (2.3) the summation is over all proper subsets R of $\{1, \ldots, N + 1\}$ and complements are in the latter set. $E^{\otimes j}$ means $E \otimes \cdots \otimes E$ (j factors). Equation (2.3) says that u_{N+1} differs from a unitary element of $\mathcal{A} \otimes \mathcal{I}_0^{\otimes N+1}$ by a linear combination of elements formed from $(u_2, \ldots, u_N)$. Provided that each $\mathcal{A} \otimes \mathcal{I}_0^{\otimes N+1}$ contains unitary elements, which will be so if $\mathcal{A}$ does, extension is always possible.

3. FLOW GENERATORS

For the linear map j: $\mathcal{A} \to \tilde{\Gamma}(\mathcal{I})$ to satisfy the relations

$$j(xy) = j(x) * j(y), \qquad j(x^\dagger) = j(x)^\dagger, \qquad x, y \in \mathcal{A}$$

corresponding to $*$-algebra morphism (flow) properties of $J = I(j)$ (if it exists), its components must satisfy

$$j_n(xy) = \sum_{R \cup S = \{1,\ldots,n\}} j_r^R(x) j_s^S(y) \tag{3.1}$$

and

$$j_n(x^\dagger) = j_n(x)^\dagger \tag{3.2}$$

for arbitrary $x, y \in \mathcal{A}$. Denote by Z the space of such maps j and by Z_N the space of *Nth-order flow generators* $(j_0, \ldots, j_N)$, where j_n: $\mathcal{A} \to \mathcal{A} \otimes \mathcal{I}^{\otimes n}$ is linear and satisfies (3.1) and (3.2) for all $n \leq N$. The extension question for such generators can again be answered affirmatively in the purely gauge case. If $(j_0, \ldots, j_N) \in Z_N$, then $(j_0, \ldots, j_{N+1}) \in Z_{N+1}$ if and only if the linear $\dagger$-map j_{N+1} satisfies

$$\begin{aligned}
&[\sum j_r^R(x)(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + j_{N+1}(x)] \\
&\quad \times [\sum j_r^R(y)(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + j_{N+1}(y)] \\
&\quad = \sum j_r^R(xy)(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + j_{N+1}(xy)
\end{aligned}$$

for arbitrary $x, y \in \mathcal{A}$, where the sum is over all proper subsets of $\{1, \ldots, N + 1\}$. Evidently there is a plentiful supply of such maps j_{N+1}, differing from unital $\dagger$-algebra morphisms by maps already determined.

4. ACTION OF EVOLUTION GENERATORS ON FLOW GENERATORS

The group G acts on the space Z by the action

$$u(j)(x) = u * j(x) * u^\dagger$$

In terms of components

$$u(j)_n(x) = \sum_{R \cup S \cup T = \{1,\dots,n\}} u_r^R j_s^S(x) u_t^{\dagger T}$$

Evidently the same formula gives an action of each G_N on Z_N.

5. EXAMPLES

Consider the first-order evolution generator $(1, u_1)$, where

$$u_1 + u_1^\dagger + u_1 u_1^\dagger = u_1^\dagger + u_1 + u_1^\dagger u_1 = 0$$

This may be extended to the element $u = (1, u_1, u_1^{\{1\}} u_1^{\{2\}}, u_1^{\{1\}} u_1^{\{2\}} u_1^{\{3\}}, \dots)$ of G for which $U = I(u)$ is the iterative solution of the stochastic differential equation

$$dU = Uu_1, \qquad U(0) = 1 \tag{5.1}$$

Existence, uniqueness, and unitarity of the solution of (5.1) were proved in Hudson and Parthasarathy (1984) in the case when the components of u_1 in $\mathcal{A}$ are norm bounded.

Similarly the first-order flow generator (id, j_1), where the †-map j_1 satisfies

$$j_1(xy) = j_1(x)y + xj_1(y) + j_1(x)j_1(y)$$

extends to the element $j = (id, j_1, (j_1 \otimes id)j_1, (j_1 \otimes id \otimes id)(j_1 \otimes id)j_1, \dots)$ of Z for which $J = I(j)$ is the quantum stochastic flow satisfying

$$dJ(x) = J \otimes id j_1(x), \qquad J_0(x) = x$$

for which an existence theorem was proved in Evans (1989) in the bounded case.

An amusing example of a different kind is found by taking the trivial initial algebra $\mathbb{C}$ and seeking a flow generator $j = (j_0, j_1, \dots)$ of the form

$$j_n(x) = xe_n \qquad (x \in \mathbb{C})$$

where $e_n = e_n^\dagger$ is an element of $\mathcal{I} \otimes \cdots \otimes \mathcal{I}^n$ (n factors). In the pure gauge case, (3.1) is satisfied if

$$e_n = (-1)^n e \otimes \cdots \otimes e$$

where e is an idempotent in $\mathcal{I}$, as is easily seen using the identity

$$(-1)^n = \sum_{R \cup S = \{1,\ldots,n\}} (-1)^r(-1)^s$$

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